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RELIABILITY ANALYSIS OF STRUCTURES WITH WEIBULL DISTRIBUTIONS OF LOAD AND STRENGTH TO A GIVEN CONFIDENCE LEVEL

by

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SUMMARY

The statistical variation of load and strength is described by a three parameter Weibull distribution. The Weibull parameters are evaluated by a least square analysis and a method is presented which allows confidence bounds to be assigned to these quantities. A Monte Carlo analysis is used to calculate the reliability of the structure from the load and strength distributions.

CONTENTS

1 INTRODUCTION

The introduction of new materials for load carrying applications often presents the designer with problems in assessing the reliability of the structure. Although considerable advances have been made by material scientists in the development of new materials capable of operating in severe environments, it is only during recent years that engineers have considered introducing probabilistic methods of design into structural analysis.

With the advent of engineering plastics, fibre reinforced composites, graphites, ceramics and high strength steels, many materials are being developed which have inherent variability in their properties. For example, most graphites and ceramics have random distributions of flaws, both through their volume and on their surface, which lead to significant strength variations between nominally identical components subjected to identical states of stress[1]. It is a rare event in the aerospace industry to find a material property or an applied load which is not subject to some significant statistical variation.

The relative strengths of most structures vary statistically and direct measurement of the strength of each member could be uneconomic and impracticable. To overcome these difficulties the structural reliability may be assessed from a mathematical analysis based on the test results from a small sample of components.

Once the statistical variations of strength and load have been approximately modelled by some distribution function, the reliability of the structure may be evaluated by analysing the interaction between the two distributions. At the expense of imposing certain penalties reliability may be improved by reducing the applied load, by raising the mean strength of the material substantially, or by reducing the variability of the material under development. If the material is made for a particular Service application strength can only be increased by thickening cross sections. This is not only expensive but may increase the weight to an unacceptable level in certain aerospace applications. Material variability can be reduced if there is a choice of materials available. The material variability of some composites can be reduced without adversely affecting the strength and other properties, by changing the material composition slightly.

The accuracy of predicting the extreme values of a statistical variable strength and load depends on the fidelity of the model, and on the amount of data available and their integrity. Predictions based on the analysis of several hundred tests may be expected to give reliable results but those

evaluated from a few tests are inevitably less precise. In these circumstances predictions are made using conservative estimates for the population parameters.

Many statistical distributions are commonly used to describe experimental data and the choice of distribution in the field of reliability is by no means unique. The normal distribution might appear to be suitable and has the advantage that the evaluation of the confidence limits for small sample data is well documented[2]. However, the normal distribution is symmetric and is inappropriate for fracture data with skewed distributions. It is also double sided, extending to infinity in both positive and negative directions, which implies a finite probability value when the distribution variate is zero. In a fracture analysis application the normal distribution would yield a finite probability value for zero load which is clearly physically inadmissible, but it is negligible in practical terms and allows the normal distribution to be used in many cases.

A distribution which is bounded to the left and allows for positive and negative skewness is the one due to Weibull[3], which has wide application in the analysis of fracture and material strength and is adopted in this analysis. A Weibull distribution is also chosen to describe the statistical variation of loads. Minor modifications would be needed if the load variations were described by some other distribution function.

In Section 2 of this report the strength and load variations are each described by a three parameter Weibull distribution function. A least square analysis is presented which allows the various Weibull parameters to be calculated from the known sample data. Confidence limits are assigned to the parameters through an analysis outlined in Section 3. In Section 4 a Monte Carlo method is presented which allows the reliability of a structure to be derived from the strength and load distributions.

2 WEIBULL ANALYSIS

When the strength capability of a structure is exceeded by an applied load the structure fails and no longer meets its original design requirement. If the variability associated with strength and load is small it is possible, to a reasonable degree of accuracy, to express failure by an inequality $Y>X$. Here X and Y denote strength and applied load respectively. There are many situations where X and Y are not deterministic quantities but vary statistically according to some distribution. In this case a criterion of failure expressed in terms of some inequality becomes inadmissible and must be replaced by an expression for probability of failure.

The Weibull distribution[3] is extensively used in engineering and design, and its application to the strength analysis of brittle materials, composites and polymers is well documented[4,5,6,7,8]. If the three parameter equation is used an allowance may be made for threshold value, skewness and scale. The threshold value defines a point in the range below which the probability of a critical event is zero. Above the threshold value the probability of a critical event conforms to the Weibull distribution.

In this analysis it is assumed that both the strength and applied loads are distributed according to the three parameter Weibull distribution. It is anticipated that in most applications this assumption will be reasonable for the strength distribution but there may be certain circumstances in which a different distribution is required to model the statistical variation of the load. In this event the analysis will require minor modification.

The cumulative probability functions for the applied load and strength are given by the respective relationships

$$\left.\begin{aligned} P_X &= 1 - \exp\left\{ - \left(\frac{X - X_T}{X_0} \right)^{M_X} \right\} , \quad X > X_T \ , \\ P_X &= 0 \ , \quad X \leq X_T \ , \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} P_Y &= 1 - \exp\left\{ - \left(\frac{Y - Y_T}{Y_0} \right)^{M_Y} \right\} , \quad Y > Y_T \ , \\ P_Y &= 0 \ , \quad Y \leq Y_T \ . \end{aligned}\right\} \qquad (2)$$

In the above equations P_X and P_Y are the cumulative probability values, M_X and M_Y are the Weibull moduli, X_0 and Y_0 are the normalising factors, X_T and Y_T are the threshold values of the respective variates X and Y. The Weibull moduli and normalising factors respectively govern the skewness and scale of the distribution. The threshold values are responsible for origin shift. In terms of these Weibull parameters the distribution means and coefficients of variation are given respectively by[9]

$$\left.\begin{aligned}\bar{X} &= X_T + X_0\,\Gamma\left(1+\frac{1}{M_X}\right) ,\\ \bar{Y} &= Y_T + Y_0\,\Gamma\left(1+\frac{1}{M_Y}\right) ,\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned}C_X &= \frac{\sqrt{\Gamma\left(1+\frac{2}{M_X}\right) - \left\{\Gamma\left(1+\frac{1}{M_X}\right)\right\}^2}}{\Gamma\left(1+\frac{1}{M_X}\right) + \frac{X_T}{X_0}} ,\\[2ex] C_Y &= \frac{\sqrt{\Gamma\left(1+\frac{2}{M_Y}\right) - \left\{\Gamma\left(1+\frac{1}{M_Y}\right)\right\}^2}}{\Gamma\left(1+\frac{1}{M_Y}\right) + \frac{Y_T}{Y_0}} .\end{aligned}\right\} \tag{4}$$

The Weibull parameters appearing in equations (1) and (2) are population values. For most engineering applications these quantities are not available but will need to be estimated from small sample data to a given confidence level. If the small sample data

$$\left.\begin{aligned}X &= X_i \quad , \quad X_{i+1} > X_i \quad ,\\ Y &= Y_i \quad , \quad Y_{i+1} > Y_i \quad ,\end{aligned}\right\} \quad i=1,2, \;\ldots.\; N \;, \tag{5}$$

are available, where for convenience the size of each sample is taken to be N, the Weibull parameters may be evaluated by a least square analysis[10]. This is achieved by minimising the error terms

$$\left.\begin{aligned}\delta_X^{(i)} &= X\left(P_X^{(i)}\right) - X_i = \hat{X}_i - X_i \quad ,\\ \delta_Y^{(i)} &= Y\left(P_Y^{(i)}\right) - Y_i = \hat{Y}_i - Y_i \quad ,\end{aligned}\right\} \tag{6}$$

with respect to the unknown Weibull parameters. In equation (6) $\hat{X}_i$ and $\hat{Y}_i$ satisfy the transformed Weibull equations

$$\hat{X}_i = X_T + X_0 \left\{ \ln\left[1/(1 - P_X(i)) \right] \right\}^{\frac{1}{M_X}} ,$$

$$\hat{Y}_i = Y_T + Y_0 \left\{ \ln\left[1/(1 - P_Y(i)) \right] \right\}^{\frac{1}{M_Y}} , \tag{7}$$

where

$$P_X(i) = P_Y(i) = (i - \tfrac{1}{2})/N . \tag{8}$$

Computations have shown that the assigned experimental probability values provided by equation (8) lead to unbiased estimates of the Weibull parameters[11].

It can be shown that the minimisation of equation (6) is equivalent to solving the least square equations

$$N X_T + X_0 \sum_{i=1}^{N} W_i^{1/M_X} = \sum_{i=1}^{N} X_i , \tag{9}$$

$$X_T \sum_{i=1}^{N} W_i^{1/M_X} + X_0 \sum_{i=1}^{N} W_i^{2/M_X} = \sum_{i=1}^{N} X_i W_i^{1/M_X} , \tag{10}$$

$$X_T \sum_{i=1}^{N} W_i^{1/M_X} \ln W_i + X_0 \sum_{i=1}^{N} W_i^{2/M_X} \ln W_i = \sum_{i=1}^{N} X_i W_i^{1/M_X} \ln W_i , \tag{11}$$

and

$$N Y_T + Y_0 \sum_{i=1}^{N} W_i^{1/M_Y} = \sum_{i=1}^{N} Y_i , \tag{12}$$

$$Y_T \sum_{i=1}^{N} W_i^{1/M_Y} + Y_0 \sum_{i=1}^{N} W_0^{2/M_Y} = \sum_{i=1}^{N} Y_i W_i^{1/M_Y} , \tag{13}$$

$$Y_T \sum_{i=1}^{N} W_i^{1/M_Y} \ln W_i + Y_0 \sum_{i=1}^{N} W_i^{2/M_Y} \ln W_i = \sum_{i=1}^{N} Y_i\, W_i^{1/M_Y} \ln W_i \quad , \tag{14}$$

where

$$W_i = \ln\left(1/(1 - P^{(i)}) \right) \quad . \tag{15}$$

Equations (9) and (10) may be solved simultaneously to express X_0 and X_T in terms of the Weibull modulus M_X . Omitting the details

$$X_0 = \frac{N \sum_{i=1}^{N} X_i\, W_i^{1/M_X} - \sum_{i=1}^{N} W_i^{1/M_X} \sum_{i=1}^{N} X_i}{N \sum_{i=1}^{N} W_i^{2/M_X} - \sum_{i=1}^{N} W_i^{1/M_X} \sum_{i=1}^{N} W_i^{1/M_X}} \quad , \tag{16}$$

$$X_T = \frac{\sum_{i=1}^{N} X_i \sum_{i=1}^{N} W_i^{2/M_X} - \sum_{i=1}^{N} W_i^{1/M_X} \sum_{i=1}^{N} X_i\, W_i^{1/M_X}}{N \sum_{i=1}^{N} W_i^{2/M_X} - \sum_{i=1}^{N} W_i^{1/M_X} \sum_{i=1}^{N} W_i^{1/M_X}} \quad . \tag{17}$$

Substitution for X_0 and X_T into equation (11) leads to a transcendental equation for M_X . Although its final form is too complex to be written in this text the resulting equation may be readily solved numerically on a high speed computer. In the root solving routine adopted a starting value of $M_X = 1.0001$ was chosen because for $M \leq 1$ the Weibull equation degenerated into a physically unrealistic form for the studies under investigation. The solution for the other parameter set M_Y , Y_0 , Y_T may be obtained similarly.

3 CONFIDENCE LIMITS

In equations (1) and (2) the Weibull parameters refer to population values. The Weibull parameters calculated from equations (9) to (15) are small sample estimates and only approach the population values as the size of the sample increases to infinity. In practice the sample size may be small and the Weibull parameters evaluated from this limited data may differ significantly from the population values. In these circumstances it is useful to be able to define, for a given level of confidence, upper and lower limits of the deviation of these parameters from the population values.

When all three Weibull parameters vary statistically the estimation of the confidence bounds requires individual attention for each new set of parameters encountered. If the population threshold values are known or may be prescribed considerable simplifications occur. The confidence limits for the remaining two Weibull parameters may be evaluated from a Monte Carlo analysis of a single system of equations and these results hold universally for all Weibull distributions.

Before proceeding further it is important to note that the least square equations may be developed from any function of the differences defined by equation (6). In particular the quantities

$$\begin{aligned} \delta_X^{(i)} &= \ln\left(X\,(P_X^{(i)}) - X_T \right) - \ln\left(X_i - X_T \right) , \\ \delta_Y^{(i)} &= \ln\left(Y\,(P_Y^{(i)}) - Y_T \right) - \ln\left(Y_i - Y_T \right) , \end{aligned} \tag{18}$$

may be used to carry out the least square analysis. With these assumptions explicit expressions for M_X , X_0 , M_Y , Y_0 may be derived in terms of the threshold values X_T and Y_T . The threshold values are found as roots of two transcendental equations. Unfortunately these values are highly sensitive to small changes in data and in many cases the calculated Weibull parameters are spurious. If the quantities X_T and Y_T are prescribed, an analysis based on equation (18) yields reliable and accurate results for all cases.

If the threshold value X_T is known a least square analysis based on equation (18) yields the parameters

$$M_X^{(S)} = \frac{N \sum_{i=1}^{N} V_i^2 - \sum_{i=1}^{N} V_i \sum_{i=1}^{N} V_i}{N \sum_{i=1}^{N} V_i \ln (X_i - X_T) - \sum_{i=1}^{N} V_i \sum_{i=1}^{N} \ln (X_i - X_T)} , \tag{19}$$

$$X_0^{(S)} = \exp \left\{ \frac{\sum_{i=1}^{N} V_i^2 \sum_{i=1}^{N} \ln (X_i - X_T) - \sum_{i=1}^{N} V_i \sum_{i=1}^{N} V_i \ln (X_i - X_T)}{N \sum_{i=1}^{N} V_i^2 - \sum_{i=1}^{N} V_i \sum_{i=1}^{N} V_i} \right\}, \tag{20}$$

where

$$V_i = \ln \ln \left[1/(1-P_X^{(i)}) \right] . \tag{21}$$

In the above equations the small sample values $M_X^{(S)}$, $X_0^{(S)}$ are introduced to distinguish them from the population values M_X , X_0 . Making the change of variable

$$x = \left(\frac{X - X_T}{X_0} \right)^{M_X} \tag{22}$$

reduces the Weibull distribution to the exponential distribution

$$P_X = 1 - \exp (-x) . \tag{23}$$

Substituting equation (22) into equation (19) leads to

$$M_X^{(S)} = \frac{N\sum_{i=1}^{N} V_i^2 - \sum_{i=1}^{N} V_i \sum_{i=1}^{N} V_i}{N\sum_{i=1}^{N} V_i\left(\frac{\ln x_i}{M_X} + \ln X_0\right) - \sum_{i=1}^{N} V_i \sum_{i=1}^{N}\left(\frac{\ln x_i}{M_X} + \ln X_0\right)} \quad . \tag{24}$$

The quantities x_i are derived from equation (22) with X set equal to X_i $i=1,2, \ldots . N$. If the terms in the brackets appearing in equation (24) are expanded,

$$\frac{M_X^{(S)}}{M_X} = \frac{N\sum_{i=1}^{N} V_i^2 - \sum_{i=1}^{N} V_i \sum_{i=1}^{N} V_i}{N\sum_{i=1}^{N} V_i \ln x_i - \sum_{i=1}^{N} V_i \sum_{i=1}^{N} \ln x_i} \quad . \tag{25}$$

When equation (22) is substituted into equation (20) and the resulting expression simplified it can be shown after some manipulation that

$$\left[\frac{X_0^{(S)}}{X_0}\right]^{M_X^{(S)}} = \exp\left\{\frac{\sum_{i=1}^{N} V_i^2 \sum_{i=1}^{N} \ln x_i - \sum_{i=1}^{N} V_i \sum_{i=1}^{N} V_i \ln x_i}{N\sum_{i=1}^{N} V_i \ln x_i - \sum_{i=1}^{N} V_i \sum_{i=1}^{N} \ln x_i}\right\} . \tag{26}$$

In the above two equations the right hand sides in each case are independent of the Weibull parameters. They depend only on the distribution of x, the random variable of the exponential distribution. This implies that a Monte Carlo analysis applied to equations (25) and (26), for various values of N, will result in distributions of $M_X^{(S)}/M_X$ and $(X_0^{(S)}/X_0)^{M_X^{(S)}}$ which are universally valid for all Weibull parameters.

Random samples of any size N may be generated by evaluating the inverse of the exponential distribution

$$x_i = -\ln(1-P_x^{(i)}) \quad , \quad i=1,2, \dots N \tag{27}$$

for a set of random numbers $P_x^{(i)}$ between zero and unity. These quantities may then be substituted into equations (25) and (26) and the values of $M_X^{(S)}/M_X$ and $(X_0^{(S)}/X_0)^{M_X^{(S)}}$ calculated for that particular sample. The quantities V_i are known and given by equations (8) and (21). If this procedure is repeated many times for the same sample size N, a typical number of times being 2500, distributions of $M_X^{(S)}/M_X$ and $(X_0^{(S)}/X_0)^{M_X^{(S)}}$ may be established. It is then a trivial matter to evaluate the confidence limits directly from the percentiles of the ranked distributions. A Monte Carlo diagram showing this sequence is given in Fig. 1.

The above procedure may be repeated for any sample size N. Considerable savings in computer time may be achieved if the results can be expressed as some equation of N and the confidence bounds. To establish such relationships 500 values of $M_X^{(S)}/M_X$ and $(X_0^{(S)}/X_0)^{M_X^{(S)}}$ were obtained 10 times for each value of N ranging from 10 to 100 in steps of 10. The first three moments of the distribution of results were evaluated to measure the mean, variance and skew. An examination of these parameters and the ordered results indicate a log-normal distribution in both cases. The variables were converted to their logarithms and subsequent analysis confirmed the distribution to be log-normal. The average values of the mean, variances and skew moments are listed in Tables 1 and 2.

In Table 1 it is shown that for each sample size both the mean and variance are approximately equal to the reciprocal of the sample size. The skew moments are small and appear to be distributed about zero indicating symmetrical distributions. The distribution of $\ln(M_X^{(S)}/M_X)$ can be approximated very closely by a normal distribution with mean and variance equal to $1/N$ implying that

$$\ln\left(\frac{M_X^{(S)}}{M_X}\right) = \frac{1}{N} + Z\frac{1}{\sqrt{N}} \tag{28}$$

where Z is the normal standard deviate corresponding to the prescribed value of the confidence limit. Confirmation of this result was obtained by a Monte Carlo analysis in which the values of $M_X^{(S)}/M_X$ were evaluated for various values of N from a particular Weibull distribution whose population parameters were $M_X = 2$, $X_0 = 1$ and $X_T = 0$. The agreement between the two methods is very close, as shown in Table 3.

The mean value of the ratio $\ln\,(X_0^{(S)}/X_0)^{M_X^{(S)}}$ does not appear to be related to the sample size, as Table 2 shows, but being smaller than its standard deviation it may be assumed to be zero. The standard deviation of the ratio has a log-linear relationship with the sample size N and may be closely approximated by the equation

$$\sigma_{X_0} = \frac{3}{2}N^{-\sqrt{1/3}} \tag{29}$$

with a correlation coefficient of 0.9987. In the above equation σ_{X_0} denotes the standard deviation of $\ln(X_0^{(S)}/X_0)^{M_X^{(S)}}$. It therefore follows that

$$\ln(X_0^{(S)}/X_0)^{M_X^{(S)}} = \frac{3}{2}N^{-\sqrt{1/3}}\,Z \tag{30}$$

where Z is the normal standard deviate corresponding to the prescribed confidence limit value. A Monte Carlo analysis gave similar results, as shown in Table 4.

4 RELIABILITY EVALUATION

Equation (7) allows the distribution variates X and Y to be expressed in terms of the Weibull population parameters. In practice these quantities are not usually known and instead only estimates from small sample values are available. Provided that the threshold values X_T and Y_T can be prescribed, the population values for the remaining Weibull parameters lie within the confidence intervals

$$M_X^{(S)} \exp\left\{-\frac{1}{N}+\frac{Z_1}{\sqrt{N}}\right\} < M_X < M_X^{(S)} \exp\left\{-\frac{1}{N}+\frac{Z_2}{\sqrt{N}}\right\} \tag{31}$$

$$X_0^{(S)} \exp\left\{\frac{3}{2M_X^{(S)}} N^{-\sqrt{1/3}} Z_1\right\} < X_0 < X_0^{(S)} \exp\left\{\frac{3}{2M_X^{(S)}} N^{-\sqrt{1/3}} Z_2\right\} \tag{32}$$

with a similar pair of equations for M_Y and Y_0 . The quantities Z_1 and Z_2 are the normal standard deviates for the lower and upper confidence limits respectively.

Failure occurs when the applied loads exceed the component strength, and the structural reliability may be obtained by considering the distribution of their differences. A conservative estimate of failure probability may be obtained by taking upper and lower confidence estimates of the population values of the respective load and strength distributions. The minimum and maximum values of the respective variates X and Y follow from substitution of the values of the inequalities given by equations (31) and (32) into equation (7). Omitting the details the minimum and maximum values of X and Y are given by

$$X = X_T + X_0^{(min)} \alpha_X^{1/M_X^{(min)}} , \tag{33}$$

$$Y = Y_T + Y_0^{(max)} \alpha_Y^{1/M_Y^{(min)}} \tag{34}$$

where

$$X_0^{(min)} = X_0^{(S)} \exp\left\{\frac{3}{2M_X^{(S)}} N^{-\sqrt{1/3}} Z_1\right\} . \tag{35}$$

$$Y_0^{(max)} = Y_0^{(S)} \exp\left\{\frac{3}{2\,M_Y^{(S)}} N^{-\sqrt{1/3}} Z_2\right\} , \tag{36}$$

$$M_X^{(min)} = M_X^{(S)} \exp\left\{ -\frac{1}{N} + \frac{Z_1}{\sqrt{N}} \right\} , \tag{37}$$

$$M_Y^{(min)} = M_Y^{(S)} \exp\left\{ -\frac{1}{N} + \frac{Z_1}{\sqrt{N}} \right\} \tag{38}$$

$$\alpha_X = \ln\left(1/(1-P_X) \right) , \tag{39}$$

$$\alpha_Y = \ln\left(1/(1-P_Y) \right) , \tag{40}$$

Since the population parameters are constant only one confidence limit value may be chosen for each parameter. The values chosen are those which minimise the strength variable in the lower probability region (i.e. $\alpha_X < 1$) and maximise the load variable in the higher probability region (i.e. $\alpha_Y > 1$). The choice of values for α_X and α_Y is reflected in the derivation of the above equations.

The probability that Y exceeds X can now be evaluated by a Monte Carlo analysis in which the values of Y and X are selected by generating random values for P_X and P_Y. The number of occasions that Y exceeds X, expressed as a ratio of the total number of trials, gives the required failure probability. If the anticipated probability values are high the method described above may be acceptable, but for low probability values the method is both inefficient and uneconomical and an anticipated probability value of 10^{-6} would require 10^{-9} trials to give moderate results.

An alternative method is to analyse the distribution of the differences defined by

$$D_i = Y_i - X_i , \quad i=1,2, \ldots . N . \tag{41}$$

Computations have shown that up to $N = 1000$ the assumption that the distribution of the differences, D_i, is normal is not rejected by the Lilliefors or Kolmogorov Smirnov tests at the 20% level of significance[12]. Under these conditions the variate D can be expressed in the form

$$D = \mu_D + Z\,\sigma_D \tag{42}$$

where μ_D and σ_D respectively denote the mean and standard deviation of the difference distribution D_i , $i=1,2, \ldots N$. The quantity Z is the normal standard deviate. The probability that Y exceeds X is found from equation (42) by first setting D equal to zero and then finding the probability value corresponding to the normal deviate

$$Z = -\frac{\mu_D}{\sigma_D} \quad . \tag{43}$$

The required failure probability value follows from the relationship

$$P[Y>X] = P[D\geq 0] = 1 - P\left[Z = -\frac{\mu_D}{\sigma_D}\right] \quad . \tag{44}$$

A flow diagram describing the Monte Carlo simulation for calculating $P[Y>X]$ is given in Fig. 2.

To assess the accuracy of the above procedures population parameters were prescribed on the assumption that the two distributions for X and Y were normal. The population parameters were chosen to give a high failure probability for $P[Y>X]$ and insignificant probability values for P_X and P_Y as X and Y approached zero. A Monte Carlo analysis was used to generate two samples of size $M = 1000$ from the population parameters characterising the above normal distributions. Each sample was subsequently divided into smaller samples of size $N = 10, 20, 20, 50, 100, 200, 500$. The failure probability, $P[Y>X]$, was first calculated for each sample size N to a specified confidence level of 95% by the non-central 't' statistic[13]. This analysis took into account the total interaction between the two distributions P_X and P_Y but was valid only for normal distributions. Weibull distributions were then fitted to the same sets of data and the small sample estimates of the Weibull parameters calculated. Using these parameters and a specified confidence level of 95% the probability of Y exceeding X , $P[Y>X]$, was evaluated by the Monte Carlo analysis outlined at the beginning of this section. A comparison between the Monte Carlo analysis and the results obtained from the non-central 't' statistic are shown in Table . Agreement is close in all cases, the Monte Carlo results being slightly more conservative.

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TABLE 1

Distribution parameters of $\ln\left(\frac{M_X^{(S)}}{M_X}\right)$

derived from a Monte Carlo analysis

Sample size	Mean	Variance	Skew moment
10	0.0881	0.1026	0.5437
20	0.0494	0.0490	0.0420
30	0.0331	0.0336	-0.009
40	0.0258	0.0258	0.2567
50	0.0187	0.0212	0.2716
60	0.0122	0.0172	0.1164
70	0.0150	0.0147	-0.0591
80	0.0132	0.0131	0.0972
90	0.0143	0.0119	0.1101
100	0.0096	0.0104	-0.1348

TABLE 2

Distribution parameters of $\ln \left[\left(\frac{X_0^{(S)}}{X_0} \right)^{M_X^{(S)}} \right]$

derived from a Monte Carlo analysis

Sample size	Mean	Variance	Skew moment
10	-0.0322	0.1712	-2.62×10^{-2}
20	-0.0210	0.0703	-1.10×10^{-3}
30	-0.0167	0.0458	1.26×10^{-4}
40	-0.0117	0.0322	3.20×10^{-5}
50	-0.0153	0.0244	-4.84×10^{-4}
60	-0.0083	0.0202	1.32×10^{-5}
70	-0.0077	0.0174	1.43×10^{-5}
80	-0.0076	0.0159	-1.85×10^{-5}
90	-0.0047	0.0139	-1.80×10^{-4}
100	-0.0003	0.0119	-6.19×10^{-5}

TABLE 3

Distribution parameters of $\ln\left(\frac{M_X^{(S)}}{M_X}\right)$ obtained by Monte Carlo methods from a Weibull distribution with $M = 2$, $X_0 = 1$, $X_T = 0$.

Sample size	Mean	Variance	$\frac{\text{4th moment}}{\text{variance}^2}$
10	0.0851	0.1007	3.20
20	0.0452	0.0492	2.98
30	0.0353	0.0325	2.92
40	0.0271	0.0252	2.97
50	0.0215	0.0195	2.97
60	0.0186	0.0166	2.93
70	0.0169	0.0143	2.90
80	0.0158	0.0131	3.06
90	0.0139	0.0113	3.00
100	0.0130	0.0103	2.98

For each value N the distributional parameters of 500 samples of the ratio $\ln\left(\frac{M_X^{(S)}}{M_X}\right)$ were obtained 10 times and averaged.

Note, for a normal distribution $\frac{\text{4th moment}}{\text{variance}^2} = 3.0$.

TABLE 4

Distribution parameters of $\ln\left[\left(\frac{X_0^{(S)}}{X_0}\right)^{M_X^{(S)}}\right]$ obtained by

Monte Carlo methods from a Weibull distribution

with $M = 2$, $X_0 = 1$, $X_T = 0$.

Sample size	Mean	Variance	$\frac{\text{4th moment}}{\text{variance}^2}$
10	-0.0197	0.1654	4.47
20	-0.0174	0.0657	3.60
30	-0.0087	0.0433	3.15
40	-0.0062	0.0315	3.27
50	-0.0035	0.0242	3.36
60	-0.0022	0.0203	3.11
70	-0.0006	0.0175	3.08
80	-0.0015	0.0158	3.12
90	-0.0008	0.0133	3.17
100	-0.0004	0.0122	3.13

For each value of N the distributional parameters of 500 samples of the ratio $\ln\left[\left(\frac{X_0^{(S)}}{X_0}\right)^{M_X^{(S)}}\right]$ were obtained 10 times and averaged.

Note, for a normal distribution $\frac{\text{4th moment}}{\text{variance}^2} = 3.0$.

TABLE 5

Probability of failure calculations for various sample sizes at a confidence level of 95%

Sample size	Probability of failure P[Y>X]	
	Non central 't' statistic	Monte Carlo analysis
10	0.3606	0.3897
20	0.2310	0.2711
30	0.2054	0.2528
50	0.2373	0.2598
100	0.1611	0.1808
200	0.1470	0.1650
500	0.1358	0.1347
1000	0.1276	0.1399

Fig. 1 Monte Carlo diagram for confidence limits

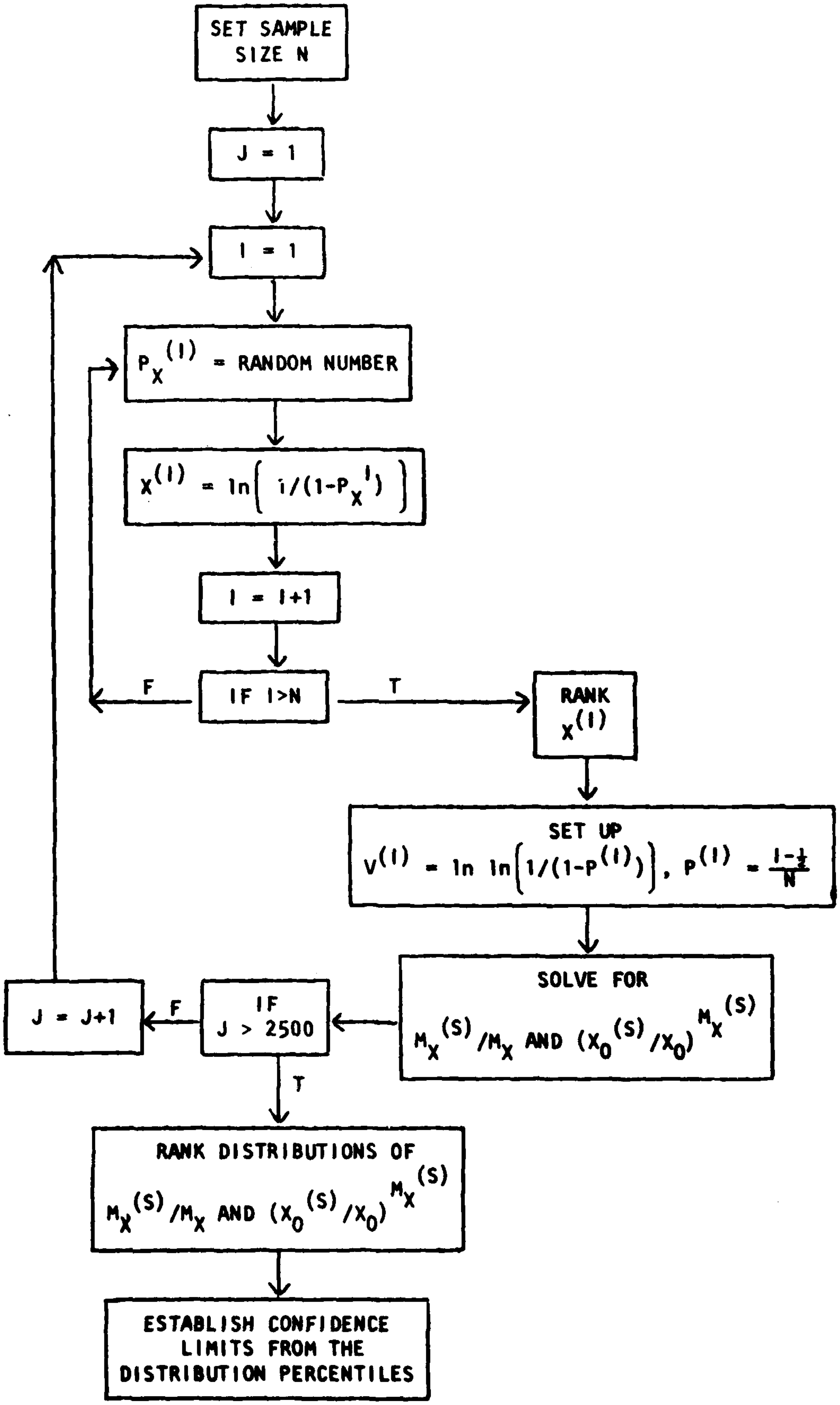
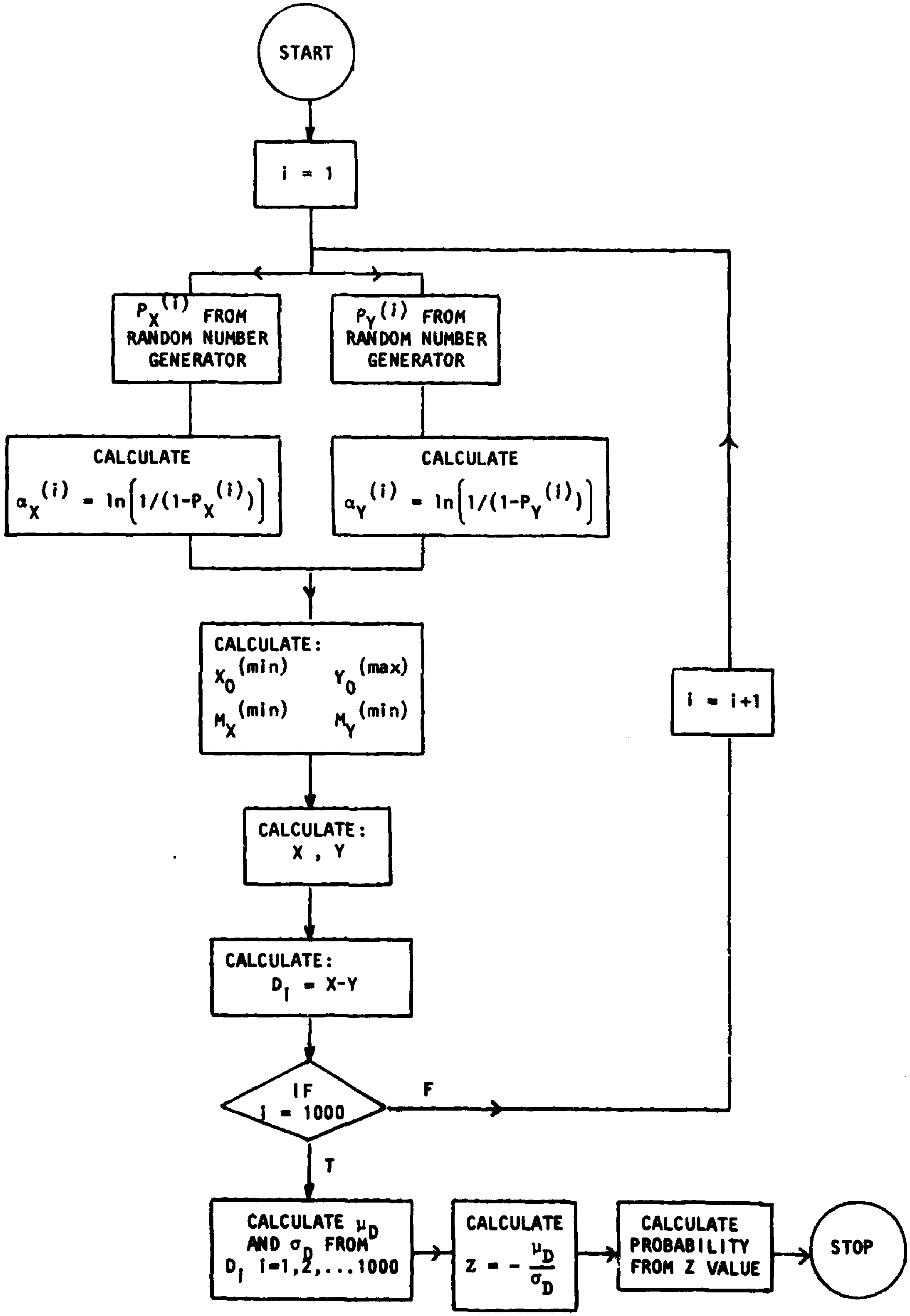


Fig. 2 Method of calculating P[Y>X] using the Monte Carlo similation



DOCUMENT CONTROL SHEET

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7. Title

RELIABILITY ANALYSIS OF STRUCTURES WITH WEIBULL DISTRIBUTIONS OF LOAD AND STRENGTH TO A GIVEN CONFIDENCE LEVEL

7a. Title in Foreign Language (in the case of translations)

7b. Presented at (for conference papers). Title, place and date of conference

8. Author 1, Surname, initials	9a Author 2	9b Authors 3, 4...	10. Date	pp	ref
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Descriptors (or keywords)

Reliability; Failure; Structural analysis; Stress analysis; Weibull density functions; Probability.

continue on separate piece of paper if necessary

Abstract

The statistical variation of load and strength is described by a three parameter Weibull distribution. The Weibull parameters are evaluated by a least square analysis and a method is presented which allows confidence bounds to be assigned to these quantities. A Monte Carlo analysis is used to calculate the reliability of the structure from the load and strength distributions.